Reasoning and Proof…A Logical Way of Thinking

Brief Overview:

In this three day unit, students will develop informal proofs. Further, students will master how to write three types of conditional statements (converse, inverse, and contrapositive) and be able to use them to describe real life situations. In combination, these concepts will lead to formal proofs later on.

NCTM Content Standard/National Science Education Standard:

- Recognize reasoning and proof as fundamental aspects of mathematics
- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs
- Select and use various types of reasoning and methods of proof

Grade/Level:

Grades 9 – 12/ Geometry

Duration/Length:

Three 45 – minute class periods

Student Outcomes:

Students will:

- Identify informal proofs
- Construct informal proofs
- Master vocabulary relating to conditional statements

Materials and Resources:

- Sentence strips
- Chessboard
- Dominoes
- Scissors
- Copies of worksheets
    - Sudoku
    - Bridge Problem
    - Taxi Cab Problem
    - Square Problem
    - Towers of Hanoi Problem
    - Number Picture Problem
    - Handshake Problem

- Revised Domino Problem
- Rubric for Group Presentations
- Logical Definitions
- Observation Record for Fishbowl Activity
- Summative Assessment

Development/Procedures:

Day 1

- Pre-assessment
  Allow the students approximately 5 minutes to solve an easy level Sudoku puzzle. (Use the puzzle provided or one from another source.) Discuss the logical thinking and overall methods students used to complete the puzzle. Ask:
  - How do students know that the number is in the correct place?
  - Why is one number put down before another?
  - Does it matter what order numbers are written in?
  - What is the connection with mathematics?

- Exploration
  Present the question, "How do you make spaghetti?" Hand out sentence strips to various students with the following statements:

Get a pot	Get a box of spaghetti
Get a jar of sauce	Get a strainer
Get a bowl	Put water in the pot
Turn on the stove heat	Boil the water
Put spaghetti in the water	Cook spaghetti
Strain the spaghetti	Put the spaghetti in the bowl
Put sauce on the spaghetti	Eat the spaghetti

  Invite students with the sentence strips to the front of the room and have students arrange themselves in order with the help of remaining seated students.

- Explanation
  Make connections between the Sudoku activity and the spaghetti activity. Explain that logical reasoning must be used to put things in the correct order and justification for the solution must be provided. Hold up a chessboard and dominoes. Pose the question: "How many dominoes would it take to cover the board?" Answer: 32 dominoes. Challenge the students by asking: "If two opposite corners are cut off, could we cover the entire board with 31 dominoes?" Conduct a Think-Pair-Share to give students the opportunity to think about the solution and discuss their thinking with a partner. Elicit responses from various pairs of students. Lead the students to the answer, if needed. Answer: No, because each

domino covers one light and one dark square and opposite corners are the same color. Be sure that students know this is an informal proof- we have shown why "no" must be correct.

Explain to the students that the discussion is considered to be an informal proof. Point out that there was given information and through logical steps, reasoning, and justification, the students showed why "no" must be the correct answer.

- Application
  Divide the class into groups (3 - 4 preferable). Give each group one of the following logic puzzles to solve and informally prove. Note: problems increase in difficulty down the list provided, allowing for differentiation within the activity. Explain to the groups that they will present their problem, solution, and justification on Day 2.

  - Revised Domino Problem
  - Square Problem
  - Handshake Problem
  - Number Picture Problem
  - Konigsberg Bridges Problem
  - Towers of Hanoi Problem
  - Taxi Cab Problem

- Differentiation
  - Reteach
    Give the Revised Domino Problem to the weakest group.

  - Enrichment
    Give the Towers of Hanoi Problem to the strongest group.

- Embedded Assessment
  Move among groups observing discussions/responses based on the problem assigned to each group.

Day 2

- Exploration
  Provide additional time for the student groups from the previous lesson consolidate their solutions and reasoning. Allow students to present their work and solutions to the various logic puzzles. Consult the "Rubric for Group Presentations" for scoring. Consider giving the rubric to the students to grade their peers.

- Explanation

After each presentation, conduct a Think-Pair-Share, giving the students an opportunity to think about the solution and discuss whether they were persuaded by the justification.

- Embedded Assessment
  For student groups, assess based on the given rubric. For class discussion, elicit understanding of what is needed to establish justification of the solution provided.

Day 3

- Exploration
  Conduct a Fishbowl activity. Divide the students into two rings- an inner discussion ring and an outer observation ring. Give students a copy of Logical Definitions and Observation Record for Fishbowl Activity.

  For first third of class time, the inner ring will discuss definitions for "Conditional Statement", "Hypothesis", and "Conclusion". The outer ring will observe the inner ring based on the Observation Record for Fishbowl Activity. Questions can be asked to prompt students to complete definitions, but teacher input should be kept to a minimum.

  For second third of class time, switch the groups. New inner ring will discuss definitions for "Converse", "Inverse", and "Contrapositive". Outer ring will observe.

- Explanation
  Refer back to Domino problem from Day 1. Conduct a Think-Pair-Share for students to write the problem as a conditional. Prompt for hypothesis (what did we start with), conclusion (what are we trying to get), and complete conditional if necessary. Example answer: If two squares are removed from a domino board, then 31 dominoes will not cover all of the squares.

- Application
  Have students write their own logic problem as a conditional statement. Have students write the converse, inverse, and contrapositive of their conditional statement.

- Differentiation
  Students with communication issues can remain in the outer circle for the entire activity.

- Assessment
  Review Observation Record for Fishbowl Activity for participation.
  Review application- be sure student groups have correctly stated their conditional, converse, inverse, and contrapostive.

Summative Assessment:

Use Summative Assessment ditto to check understanding of informal proof and vocabulary. Be sure to review significant areas of student weakness, if needed.

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__	__	5	__	2	9	__	__	__
__	7	__	8	__	__	__	__	__
5	__	3	7	__	2	9	__	4
__	__	__	__	__	1	__	8	__
__	__	__	9	1	__	6	__	__
4	__	__	__	8	3	2	__	__
__	6	__	__	__	__	__	4	__

The Seven Bridges of Konigsberg

In 1738, the city officials of Konigsberg were planning a parade to celebrate the 200th birthday of the city. They wanted the parade to cross each bridge once and only once. Were they able to have their desired parade? Provide justification for your group's response.

Konigsberg

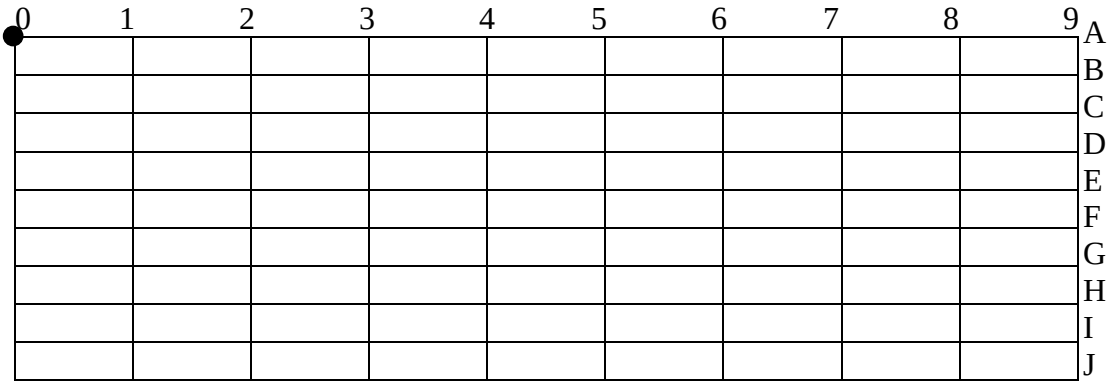
Taxi Cab Travel

Professor Fleet wants to visit his friend, Dr. N. Place. The city is designed as a grid of horizontal lines (A – J) and vertical lines (0 – 9). Professor Fleet will take a taxi from his home at (A, 0) to Dr. N. Place.

Determine how many possible routes of travel Professor Fleet can take to Dr. N. Place such that no road is used twice.

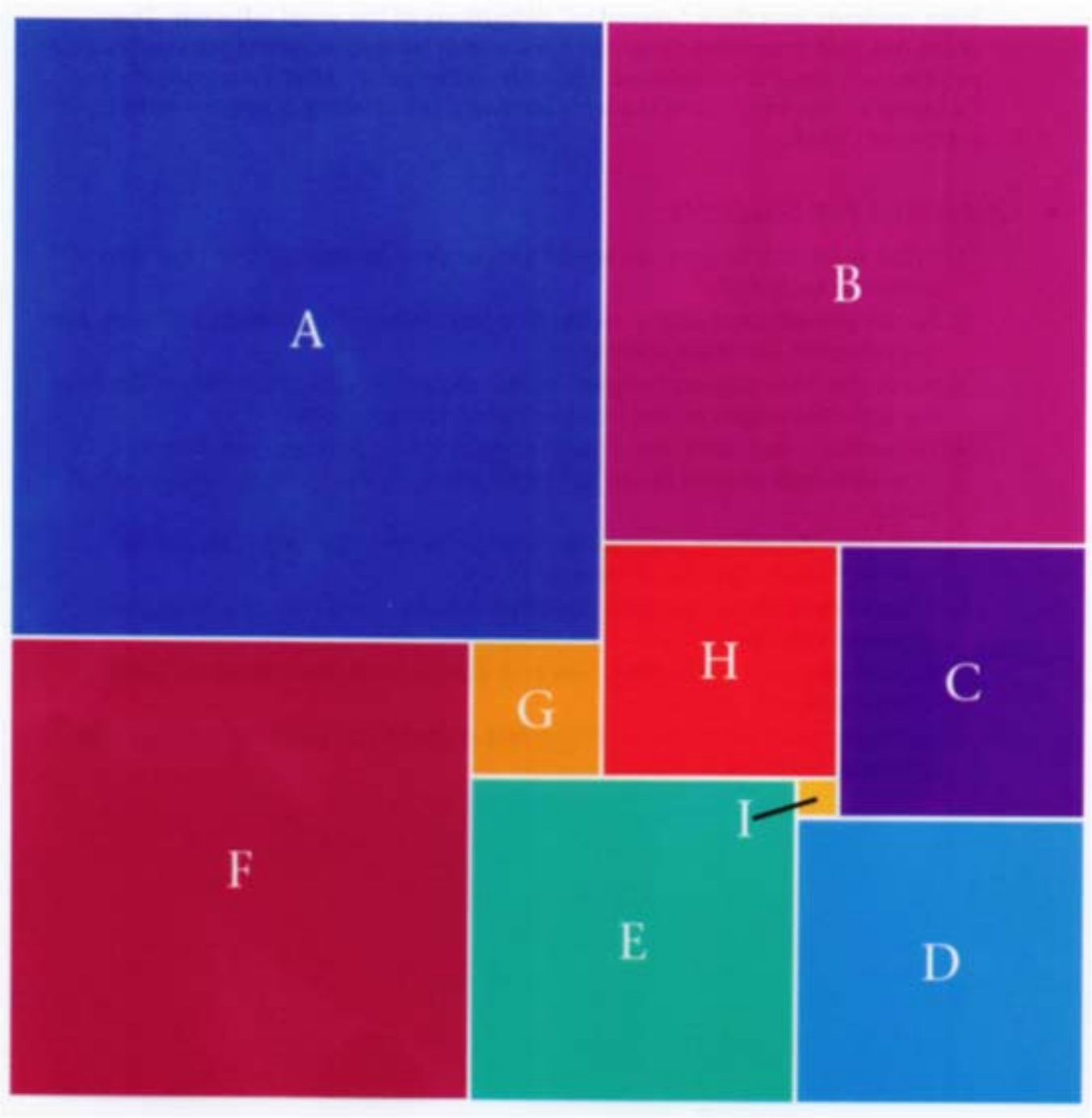
Determine a general formula so Professor Fleet knows how many routes he will have based on Dr. N. Place's location. Justify your formula.

Dr. N. Place's Location	(B, 1)	(C, 2)	(D, 3)	(E, 4)	(N, n)
Number of Routes					



The Square Problem

A figure is given below. It is made up of squares. Two facts are known about the squares- Square C has an area of 64 square units and Square D has an area of 81 square units. Determine if the overall figure is a square. Justify your answer.



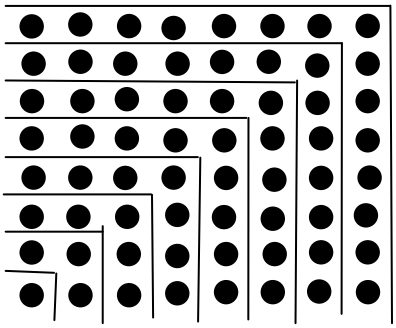
Revised Domino Problem

Latisha insists that she can cut any two squares from a chessboard and still cover the board with 31 dominoes. Michael doesn't believe her. Determine who is correct and justify your response.

Picture Number Problem

For each level of the figure below, dots are added to create a new border. The first level has 1 dot. The second level adds three dots and creates a larger figure. Determine a formula for finding the number of dots on level n. Justify your formula.

Level	1	2	3	4	5	n
# of Dots	1	4				



Handshake Problem

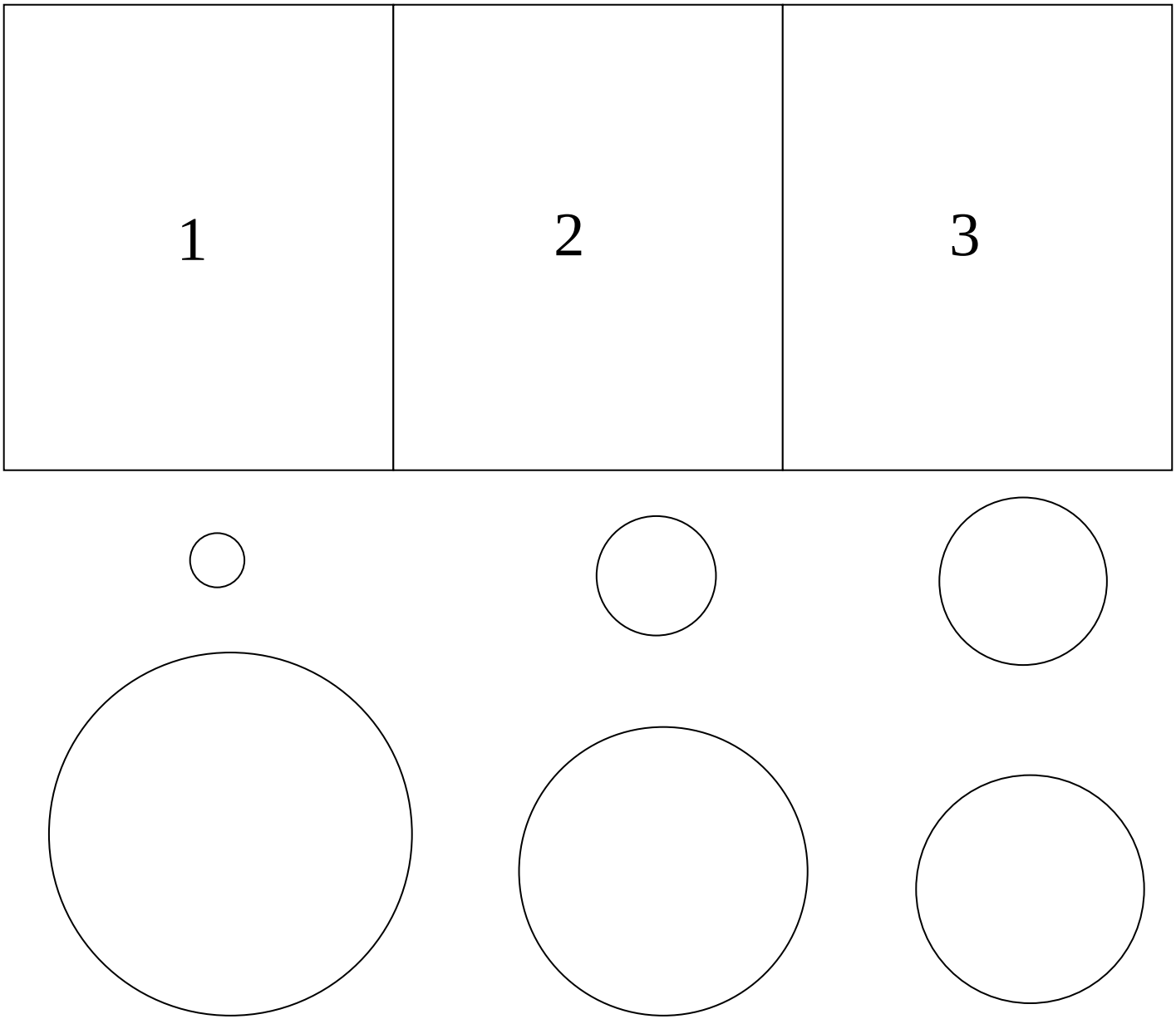
Lydia went to a party. She shook the hand of every person there. Later, she wondered how many handshakes had been exchanged at the party by everyone. Assume only two people shake hands at a time and that makes one handshake. Determine a formula for Lydia to use. Justify your formula.

Number of People (including Lydia)	2	3	4	5	6	7	n
Number of Handshakes							

Towers of Hanoi Problem

In an ancient city in India, so the legend goes, monks in a temple have to move a pile of 64 sacred disks from one location to another. There's a game based on the legend. You have a small collection of disks and three piles into which you can put them. Start all the disks in the rightmost pile and move them to the leftmost pile. There are two rules- only one disk can be moved at a time and a smaller disk cannot be under a larger disk. Determine the smallest number of moves necessary to move n disks. Justify your response.

Number of Disks	1	2	3	4	5	6	n
Number of Moves							



Rubric for Group Presentations

Score 3

The response demonstrates an understanding of the complexities of the problem.

Solution is clearly presented.

Justification for solution is explicit.

Group is able to field questions about their solution.

Score 2

The response demonstrates a general understanding of the problem.

Solution is presented.

Justification is present, but not fully explained.

Questions fielded by group are partially answered.

Score 1

The response demonstrates a minimal understanding of the problem.

A partial solution is presented.

Justification is exceedingly vague or non-existent.

Group cannot answer questions about their problem.

Score 0

The response is completely incorrect, irrelevant to the question, or missing.

Note: This Rubric grade is based on MSDE BCR Rubric.

OBSERVATION RECORD FOR FISH BOWL ACTIVITY

Name_____ Person You Observe_____

EACH TIME YOU OBSERVE AN ACTIVITY BY YOUR PARTNER, CHECK THE APPROPRIATE BOX

COMMENTS MADE:

COMMENTS USING TEXT/NOTES:

INAPPROPRIATE/NEGATIVE COMMENTS:

LOOKING AT PERSON WHO IS SPEAKING:

ENGAGING IN SIDE CONVERSATIONS:

1. What was the most interesting thing your partner said?

2. What would you have added to the conversation?

3. What questions do you still have about the topic?

Logical Definitions

Use the following examples to create a definition for each term below.

Conditional Statement:

Conditional Statements	Not Conditional Statements
1. If I study hard, then I will get a good grade.	I will study hard to get a good grade.
2. If you love me, then you set me free.	I will set you free because I love you.
3. If a number is 2, then it is a prime number.	Two is a prime number.

Hypothesis:

Hypothesis	Not Hypothesis
1. I study hard	I will get a good grade.
2. You love me	If you love me.
3. A number is 2.	Then it is a prime number.

Conclusion:

Conclusion	Not Conclusion
1. I will get a good grade.	I study hard.
2. You set me free.	Then you set me free.
3. It is a prime number.	If a number is 2.

Converse:

Converse	Not Converse
1. If I will get a good grade, then I study hard.	I will study hard to get a good grade.
2. If you set me free, then you love me.	If you don't love me, then set me free.
3. If a number is a prime number, then it is 2.	If it is 2, then it is prime.

Inverse:

Inverse	Not Inverse
1. If I do not study hard, then I will not get a good grade.	I will not study hard to get a bad grade.
2. If you do not love me, then you do not set me free.	I will not set you free because I love you.
3. If a number is not 2, then it is not a prime number.	Two is not a prime number.

Contrapositive:

Contrapositive	Not Contrapositive
1. If I will not get a good grade, then I do not study hard.	If I will study hard, I get a good grade.
2. If you do not set me free, then you do not love me.	If you don't love me, then set me free.
3. If a number is not a prime number, then it is not 2.	If it is 2, then it is prime.

SUMMATIVE ASSESSMENT

1. In your own words, what is an informal proof?

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2. Give an example of a situation requiring justification.

_____

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Given the conditional statement: If it rains, then the ground is wet. Label the following as converse, inverse, or contrapositive.

3. If the ground is not wet, then it does not rain. _____

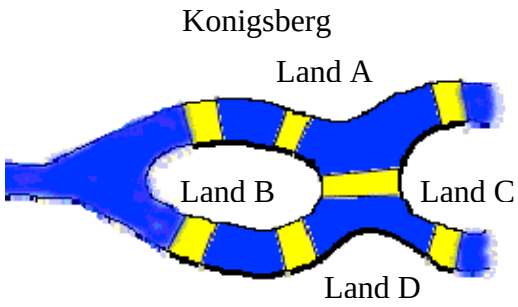
4. If the ground is wet, then it rains. _____

5. If it does not rain, then the ground is not wet. _____

8	**9**	**2**	**5**	**7**	**6**	**4**	**3**	**1**
7	1	6	3	4	8	5	2	9
3	4	5	1	2	9	8	7	6
1	7	4	8	9	5	3	6	2
5	8	3	7	6	2	9	1	4
6	2	9	4	3	1	7	8	5
2	3	7	9	1	4	6	5	8
4	5	1	6	8	3	2	9	7
9	6	8	2	5	7	1	4	3

The Seven Bridges of Konigsberg

In 1738, the city officials of Konigsberg were planning a parade to celebrate the 200th birthday of the city. They wanted the parade to cross each bridge once and only once. Were they able to have their desired parade? Provide justification for your group's response.



No, town officials could not hold their desired parade. For each land mass, there must be two bridges- one to get onto the land and one to leave the land. For the 4 sections of Konigsberg, there must be eight bridges.

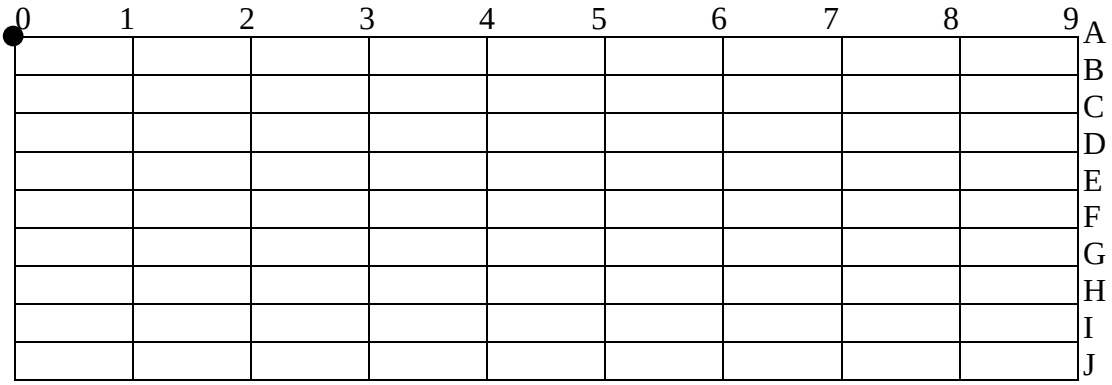
Taxi Cab Travel

Professor Fleet wants to visit his friend, Dr. N. Place. The city is designed as a grid of horizontal lines (A – J) and vertical lines (0 – 9). Professor Fleet will take a taxi from his home at (A, 0) to Dr. N. Place.

Determine how many possible routes of travel Professor Fleet can take to Dr. N. Place such that no road is used twice.

Determine a general formula so Professor Fleet knows how many routes he will have based on Dr. N. Place's location. Justify your formula.

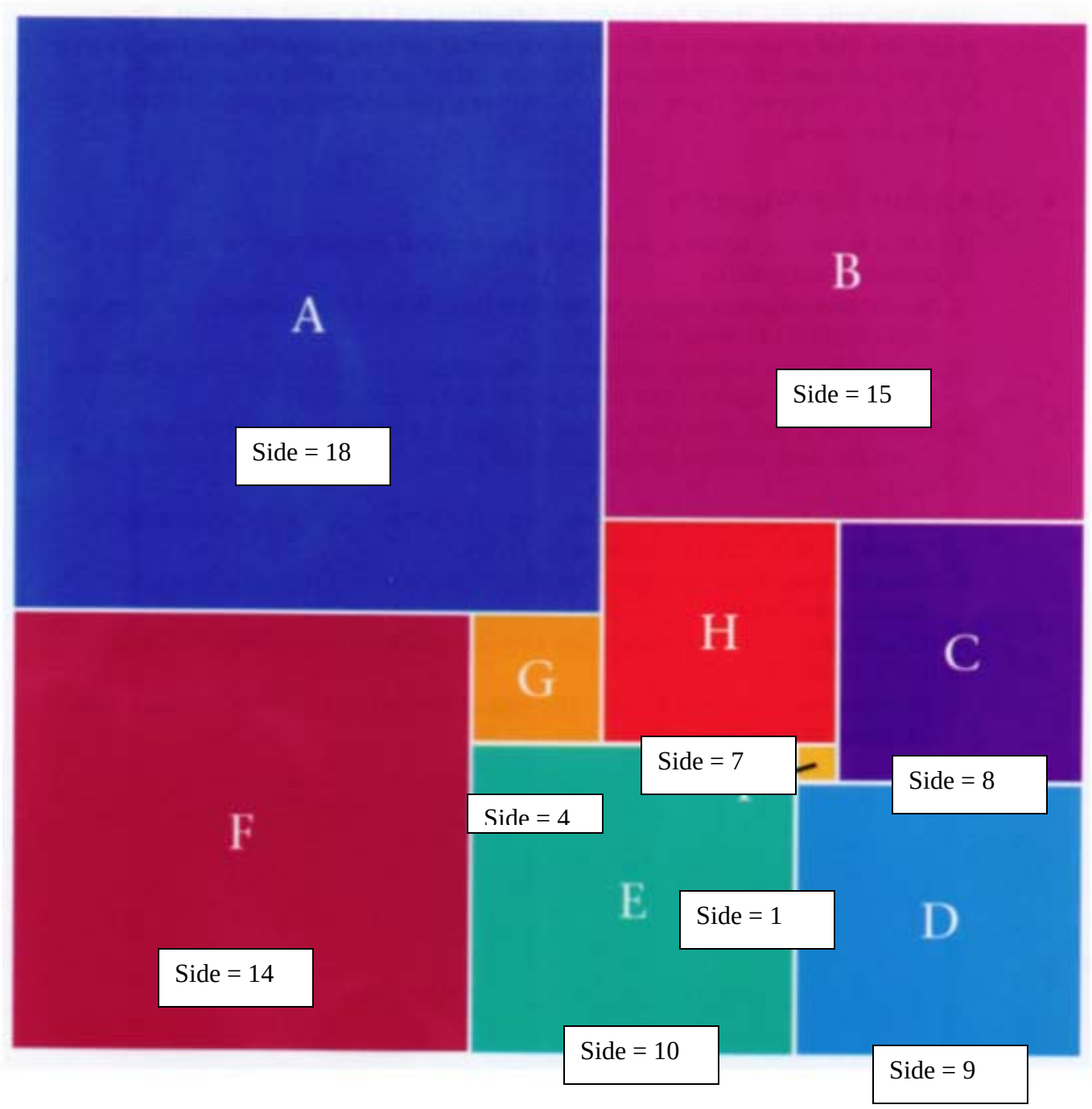
Dr. N. Place's Location	(B, 1)	(C, 2)	(D, 3)	(E, 4)	(N, n)
Number of Routes	2	6	20	70	



For any square box created, the number of routes for Professor Fleet will follow the formula $\frac{r!}{(r/2)!(r/2)!}$, where r is the sum of the lengths of the sides. The length of each side determines the number of choices available to Professor Fleet as he moves. Each move reduces the number of choices by 1 causing a factorial. The denominator reflects that we have a square box and the sum is ½ of the total length of the sides. This can be modified to deal with non-square routes.

The Square Problem

A figure is given below. It is made up of squares. Two facts are known about the squares- Square C has an area of 64 square units and Square D has an area of 81 square units. Determine if the overall figure is a square. Justify your answer.



Students will work through algebraically to find the side length of each square. Adding up the sides, the horizontal measurement is 33 units and the vertical measurement is 32 units. The figure is not a square.

Revised Domino Problem

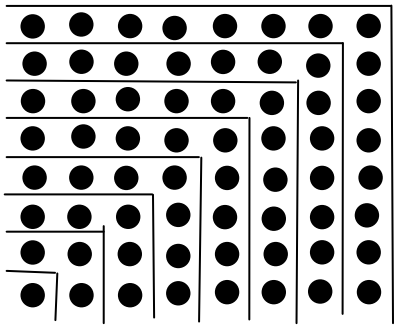
Latisha insists that she can cut any two squares from a chessboard and still cover the board with 31 dominoes. Michael doesn't believe her. Determine who is correct and justify your response.

Michael is correct. Latisha did not specify that she would cut out one light and one dark square. Because each domino covers one light and one dark square, one of each must be cut out in order to cover the chessboard.

Picture Number Problem

For each level of the figure below, dots are added to create a new border. The first level has 1 dot. The second level adds three dots and creates a larger figure. Determine a formula for finding the number of dots on level n. Justify your formula.

Level	1	2	3	4	5	n
# of Dots	1	4	9	16	25	n^2



The picture shows that we are creating a square each time we move to a new level. The number of dots is the area of the square on that level. For a square, $A = s^2$, so there will be n^2 dots on level n.

Handshake Problem

Lydia went to a party. She shook the hand of every person there. Later, she wondered how many handshakes had been exchanged at the party by everyone. Assume only two people shake hands at a time and that makes one handshake. Determine a formula for Lydia to use. Justify your formula.

Number of People (including Lydia)	2	3	4	5	6	7	n
Number of Handshakes	1	3	6	10	15	21	$\frac{n(n-1)}{2}$

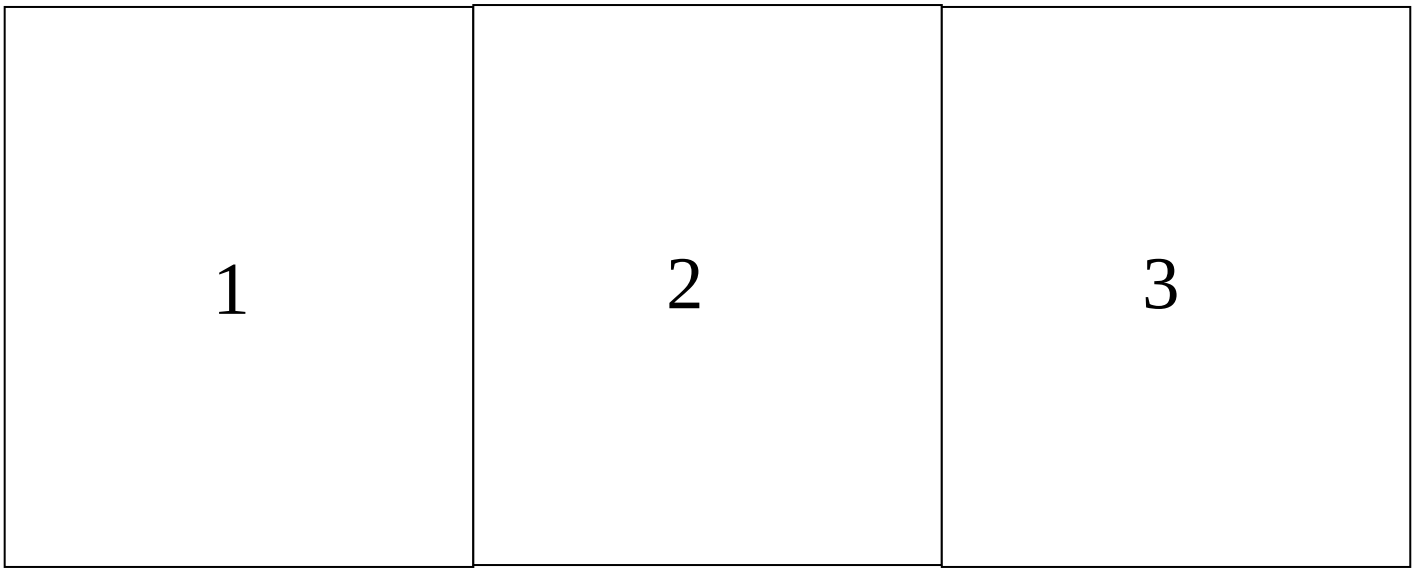
Representational drawings are highly recommended in solving this problem.
Each person (n) is shaking the hand of everyone who isn't them (n – 1). But this leads to double counting, so we must divide by 2.

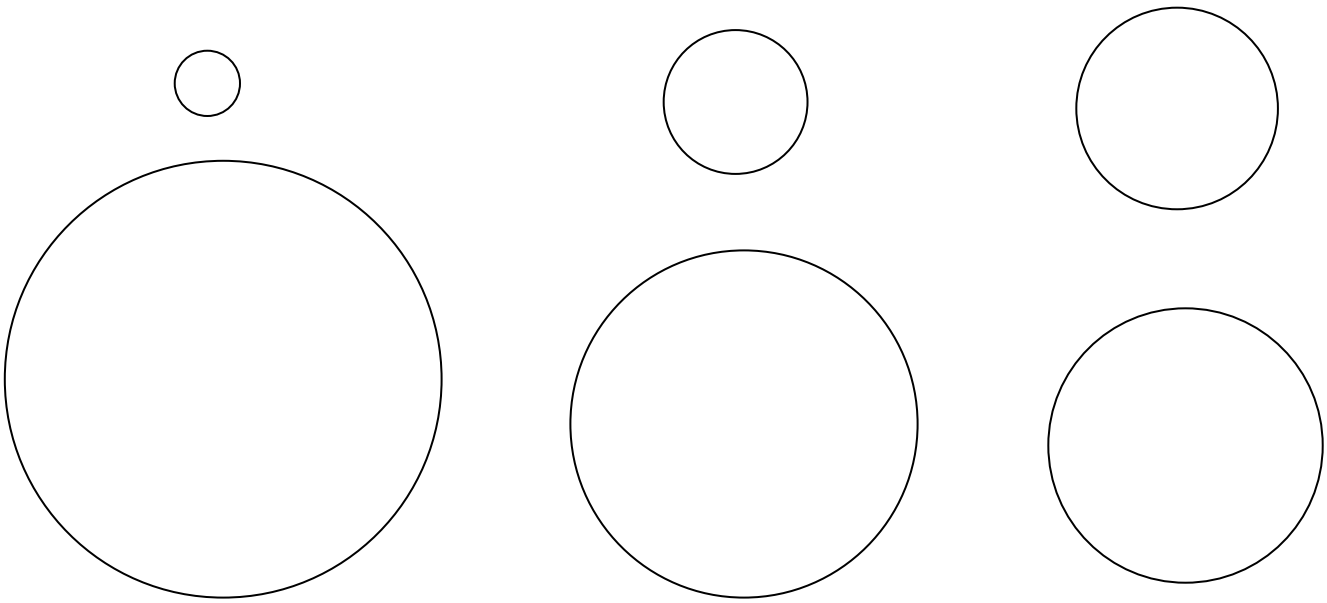
Towers of Hanoi Problem

In an ancient city in India, so the legend goes, monks in a temple have to move a pile of 64 sacred disks from one location to another. There's a game based on the legend. You have a small collection of disks and three piles into which you can put them. Start all the disks in the rightmost pile and move them to the leftmost pile. There are two rules- only one disk can be moved at a time and a smaller disk cannot be under a larger disk. Determine the smallest number of moves necessary to move n disks. Justify your response.

Number of Disks	1	2	3	4	5	6	n
Number of Moves	1	3	8	15	31	63	$2^n - 1$

Each disk has 2 choices when it is moved (2^n). Because the largest disk is only moved once to the end, we will subtract 1.





SUMMATIVE ASSESSMENT

1. In your own words, what is an informal proof?

Ex: An informal proof justifies a conclusion. It is a logical argument that makes a definite point. _____

_____

_____

2. Give an example of a situation requiring justification.

Ex: Arguing a court case. Justifying to parents why student must see a certain movie with friends. Making a claim for working a part-time job. _____

_____

_____

_____

Given the conditional statement: If it rains, then the ground is wet. Label the following as converse, inverse, or contrapositive.

3. If the ground is not wet, then it does not rain. Contrapositive

4. If the ground is wet, then it rains. Converse

5. If it does not rain, then the ground is not wet. Inverse